Applying Radiative Corrections to Polarisation Asymmetries for Deeply Inelastic Scattering

Andy Miller

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This Note presents a method to apply corrections for smearing and radiative effects, and some effects of acceptance, in the process of extraction of Born polarisation asymmetries from inclusive DIS data that has no precise absolute cross section normalization. This approach produces an unfolded model-independent result for A_{Born} in one step without the need for for model-dependent fitting with iteration. Unfolding implies the elimination of the correlations between the results in different kinematic bins caused by the smearing and radiative effects, at the cost of introduced correlations between the uncertainties in different kinematic bins. Particular attention is paid to the propagation of experimental uncertainties.

The Appetizer

First we try to provide insight by considering a simplified example where we neglect the complication of radiative DIS — by assuming there is only (quasi)-elastic radiative background. This avoids the complexities of kinematic migration and the need for the unknown result as an input to the simulations.

Let's say that, for some arbitrary MC luminosity (the same in all cases) and 100% beam and/or target polarization, we calculate with final kinematic cuts the following simulated yields in one kinematic bin:

dis_u: the unpolarized DIS signal (from the F_2 model)
bg_u: the unpolarized radiative background (from Bethe-Heitler)
bg_p: the polarized radiative background yield difference ($bg^- - bg^+$) (Bethe-Heitler has an intrinsic asymmetry)

and let's call

dis_p: the unknown polarized DIS signal yield difference ($dis^- - dis^+$) that could be simulated the same way if we knew g_1

The values for bg_p are calculated only as absolute cross sections. These must be normalized to the data by means of the combination of dis_u and bg_u. For this purpose, let's call k the (unknown) experimental efficiency×luminosity-factor for each of *both* spin states, so if the numbers of counts recorded in some bin are N^+ and N^-:[1]

$$
\begin{aligned}
(N^- + N^+) &= k(dis_u + bg_u) \quad (1)\\
(N^- - N^+) &= Pk(dis_p + bg_p), \quad (2)
\end{aligned}
$$

where P is the (product of) experimental beam and/or target polarizations, depending on whether a one-spin or two-spin asymmetry is measured. So we have 2 unknowns — dis_p and k — and 2 equations.

Immediately we find that we can calculate k:

$$k = \frac{N^- + N^+}{dis_u + bg_u} \quad (3)$$

and then

$$dis_p = \frac{N^- - N^+}{kP} - bg_p = (dis_u + bg_u)\frac{A_{\text{meas}}}{P} - bg_p \quad (4)$$

where as usual

$$A_{\text{meas}} = \frac{N^- - N^+}{N^- + N^+} \quad (5)$$

If we neglect effects of acceptance and smearing, $A_{\text{Born}} = dis_p/dis_u$, so we just divide Eq. 4 by the known dis_u to get

$$A_{\text{Born}} = \frac{dis_u + bg_u}{dis_u}\frac{A_{\text{meas}}}{P} - \frac{bg_p}{dis_u}. \quad (6)$$

[1]These yields from one measurement really represent only *estimators*. To be pedagogically correct in these equations, we should replace these with the expectation or central values of the distributions: $\langle N^\pm \rangle$.

With this formulation, it's easy to write the absolute statistical uncertainty in A_{Born}:

$$\delta(A_{\mathrm{Born}}) = \delta(dis_p/dis_u) = \frac{dis_u + bg_u}{dis_u}\frac{\delta(A_{\mathrm{meas}})}{P} \tag{7}$$

where again as usual

$$\delta(A_{\mathrm{meas}}) = \frac{2\sqrt{(N^+\delta(N^-))^2 + (N^-\delta(N^+))^2}}{(N^- + N^+)^2} \tag{8}$$

So we see statistical uncertainty inflation by the 'radiative dilution factor'

$$\frac{dis_u}{dis_u + bg_u} < 1$$

Note that it inflates regardless of whether the radiative background has the same asymmetry as the DIS signal. Note also that in this simple case we don't have to iterate, as no input is defined in terms of the unknowns.

There is also a related but different inflation of the contributions to the *systematic* uncertainty by the beam and/or target polarizations. Differentiating Eq. 6, we find

$$|\partial A_{\mathrm{Born}}| = \frac{dis_u + bg_u}{dis_u}\frac{A_{\mathrm{meas}}}{P}\frac{\partial P}{P} \tag{9}$$

$$= (A_{\mathrm{Born}} + \frac{bg_p}{dis_u})\frac{\partial P}{P}. \tag{10}$$

It is apparent that the fractional uncertainty contribution to A_{Born} is the same as that on P itself only if the radiative background has a negligible polarized cross section. Note that it is not the usual asymmetry of the background that matters — the scale is set in the denominator by the unpolarized *dis* cross section, which can be smaller than the unpolarized background at the smallest x values. Further note that even if A_{Born} is zero, it can sustain a substantial systematic uncertainty from the beam and/or target polarization.

The Main Course

Now, let's consider the effect of additional deeply inelastic radiative components. Here we have to account for kinematic migration of the DIS signal by both radiative and instrumental smearing effects.

It is assumed that there is available a Monte Carlo simulation of the detector that also includes all internal radiative effects. For each simulated event, we require both the Born and observed kinematic quantities. Using this information, we accumulate for each of the + and − spin states the two-dimensional $n_X \times (n_B+1)$ matrices $n_\pm(i,j)$ as the number of events generated that fall in both bin j of the 'true' or *Born* kinematic variables selected by the event generator to characterize the hard virtual photon, and bin i of *eXperimental* kinematic variables, which reflect radiative and instrumental effects. The indices identify only the kinematic bins, which can involve any number of kinematic variables, including those of a produced hadron. Of course, the Born binning can not be finer or involve more variables than the experimental binning, and typically may have to be coarser.

The extra bin $j = 0$ is reserved here for generated background (called *bg* in the previous section). The Born cross section in kinematic bins outside the Hermes acceptance can feed bins inside the acceptance through radiative and instrumental smearing effects. Since this contamination is independent of the unknowns inside the acceptance[2], it is included in the $j = 0$ row together with the (quasi-)elastic Bethe-Heitler contaminations.

The question may arise of which event weights to use in the accumulation of the matrices $n_\pm$, since both true (Born) and observed kinematic variables are involved.[3] It is essential that the used weights *include* the factor for radiative effects, as the smearing matrix definition obviously depends on the validity of the *observed* distributions.

For convenience in the discussion, we isolate the known unpolarized and unknown polarized distributions, defined as

$$n_u \equiv n_- + n_+ \tag{11}$$

$$n_p \equiv n_- - n_+ \tag{12}$$

The original Born distributions $n^B_\pm(j)$ defined by the MC generator can be extracted from another MC run with the *same* luminosity, with all radiative and instrumental effects (including geometric acceptance) turned off. It is important to note that

$$n^B_\pm(j) \neq \sum_{i=1}^{n_X} n_\pm(i,j), \tag{13}$$

[2]If the model for g_1 outside the acceptance needs adjustment to keep it continuous with the result inside, then an iteration may still be needed.

[3]Thanks to Juergen Wendland for raising this issue.

both because radiative effects do not conserve the total DIS cross section, and because of the events that are undetectable due to either to radiative kinematic migration outside the acceptance or to simulated detector acceptance and inefficiency. We will also require the projection of the $n_\pm$ matrices on the axis representing the eXperimental kinematics, which can be found by summing rows rather than columns:

$$n^X_\pm(i) = \sum_{j=0}^{n_B} n_\pm(i,j) \tag{14}$$

From the MC matrices $n_\pm$, we can easily calculate the *Smearing Matrices* $S_\pm(i,j)$, defined as

$$S_\pm(i,j) \equiv \frac{\partial \sigma^X_\pm(i)}{\partial \sigma^B_\pm(j)} = \frac{\partial n^X_\pm(i)}{\partial n^B_\pm(j)}. \tag{15}$$

Since radiative effects do not conserve the total cross section, the S matrices are not unitary, and their elements do not represent probabilities. If only cross sections and not amplitudes are involved in the radiative calculation, these first derivatives are constant — i.e. the higher derivatives are negligible. Then we may calculate

$$S_\pm(i,j) = \frac{n_\pm(i,j)}{n^B_\pm(j)}, \quad i = 1 \ldots n_X,\ j = 0 \ldots n_B. \tag{16}$$

The $+$ and $-$ matrices may be similar, but they may differ due to e.g. the spin-dependence of radiative effects.

The unpolarized Born distributions $B_u(j) = n^B_u(j) \equiv n^B_-(j) + n^B_+(j)$ are known from previous experiments and are incorporated in the Monte Carlo production, but only an initial guess can be used for the unknown polarized Born distribution $n^B_p(j), j = 1 \ldots n_B$. However, a key property of the smearing matrices is that they are insensitive to the model for the Born distributions used in the MC event generator. In the definition of S, both the numerator and all terms in the denominator scale together with the relative number of events generated in that Born j-bin, provided that there are no neglected kinematic variables upon which they depend differently, and upon which the acceptance imposes a bias. Hence S is independent of that number, and so it is reasonable to assume that S applies also to the real world, to the degree that the MC simulation correctly represents radiative and instrumental effects. On the other hand, if we were tempted to try to extract

an Un-smearing matrix directly from the Monte Carlo:

$$U(j,i) = \frac{n(i,j)}{n^X(i)}, \tag{17}$$

we would find it to be very sensitive to the event generator.

It is natural to wonder if a smearing matrix can be defined for unpolarized initial states. Indeed it can in an analogous manner:

$$S_u(i,j) = \frac{n_u(i,j)}{n_u^B(j)} = \frac{n_-(i,j) + n_+(i,j)}{n_-^B(j) + n_+^B(j)}, \tag{18}$$

but this concept is not useful for spin-dependent analyses, as it is in general sensitive to the event generator. This is because the relative magnitude of the two terms in the numerator can differ from their relative magnitude in the denominator, so that the above scaling argument does not work separately for the two helicity states. Such a defect is shared by a smearing matrix for any initial state that can be represented as a linear combination of two other distinct initial states.

From Eqs. 14 and 16, we have

$$n_\pm^X(i) = \sum_{j=0}^{n_B} S_\pm(i,j)\; n_\pm^B(j). \tag{19}$$

If the MC simulation really describes reality, then the observed eXperimental yields $X_\pm(i)$ can be similarly expressed in terms of the actual Born distributions $B_\pm(j)$:

$$\begin{aligned} X_-(i) &= k(i) \sum_{j=0}^{n_B} S_-(i,j)\; B_-(j), \quad i = 1 \ldots n_X & (20) \\ X_+(i) &= k(i) \sum_{j=0}^{n_B} S_+(i,j)\; B_+(j), \quad i = 1 \ldots n_X, & (21) \end{aligned}$$

or, in matrix form,

$$\begin{aligned} X_- &= [k] S_- B_- & (22) \\ X_+ &= [k] S_+ B_+, & (23) \end{aligned}$$

where k is a vector of arbitrary normalisation constants that incorporate integrated luminosities of experiment and Monte Carlo, unsimulated detector

inefficiencies that have no additional j-dependence, etc. For convenience in matrix notation, it can be expressed in the form of an $n_X \times n_X$ diagonal matrix $[k]$. We normalize the MC to the experimental results so that $B_u(j) = n_u^B(j)$, while $B_p(j), j = 1 \ldots n_B$ are the sought unknowns. We can find these together with the unknown vector k by solving the 2 vector equations, Eqs. 22 and 23, each consisting of n_X scaler equations. If this system can be solved for B_p, it provides access to the Born polarization asymmetry A_{Born} without iteration. This result would incorporate corrections for acceptance and instrumental smearing, and should also have the advantage of avoiding the smoothing effect of the fitting model for g_1 in the MC generator.[4]

As in the first section of this Note, the first step toward a solution is to use Eq. 22 to calculate k from the known unpolarized Born distribution $B_u(j)$. In the range $j = 1 \ldots n_B$ of interest, this is calculable from world data on F_2, while $B_\pm(0)$ (actually $S_\pm(i,0)\ B_\pm(0)$) must be derived from a radiative calculation, all with the same (but arbitrary) normalization. In practice, all of these known quantities are typically embodied in the same MC event generator used to define the smearing matrices; generically, $S(i,j)\ B(j) = n(i,j)$. Thus, adding Eqs. 20 and 21:

$$k(i) = \frac{X_u(i)}{\sum_{j=0}^{n_B}[S_-(i,j)\ B_-(j)\ +\ S_+(i,j)\ B_+(j)]} \tag{24}$$

$$\text{or} \quad [k](i,i) = \frac{X_u(i)}{[S_-B_-](i) + [S_+B_+](i)} \tag{25}$$

$$\text{(typically)} \quad k(i) = \frac{X_u(i)}{n_u^X(i)}. \tag{26}$$

If all detector inefficiencies that are not accounted for in the MC simulation were known to be sufficiently uniform over the kinematic range of the data (a necessary condition for which is that the $k(i)$'s are consistent within statistics), we could average all the k's, and reduce the uncertainty. This assumption may be checked more precisely by using a separate unpolarized data set with high statistics.

We now take the difference of Eqs. 20 and 21, and substitute our solution for k. As already mentioned, the $B_\pm(0)$ values representing Born processes

[4]This smoothing effect arises in the iterative method of solving for the Born observables. This effect might help explain why our previously underestimated error bars didn't result in data that appeared excessively scattered.

outside the acceptance are assumed to be available from e.g. radiative calculations (typically in the form $S_\pm(i,0)\ B_\pm(0) = n_\pm(i,0)$), with the same normalization as the unpolarized distribution ($B_u(j)$, $j = 1\ldots n_B$). Hence such a known term is moved to the other side of the equation:

$$\begin{aligned}
\textstyle\sum_{j=1}^{n_B}&[S_-(i,j)B_-(j) - S_+(i,j)B_+(j)] \\
&= \tfrac{X_p(i)}{k(i)} - S_-(i,0)B_-(0) + S_+(i,0)B_+(0), \quad i = 1\ldots n_X && (27)\\
&= A_\mathrm{X}(i)\textstyle\sum_{k=0}^{n_B}[S_-(i,k)B_-(k) + S_+(i,k)B_+(k)] \\
&\qquad -S_-(i,0)B_-(0) + S_+(i,0)B_+(0) && (28)\\
&= A_\mathrm{X}(i)X_u(i) - S_-(i,0)B_-(0) + S_+(i,0)B_+(0) && (29)\\
\text{(typically)} &= A_\mathrm{X}(i)\ n_u^X(i) - n_p(i,0), \quad i = 1\ldots n_X, && (30)
\end{aligned}$$

where

$$\begin{aligned}
B_-(j) + B_+(j) &\equiv B_u(j) = n_u^B(j), \quad j = 1\ldots n_B, && (31)\\
\text{and}\quad A_\mathrm{X}(i) &= \frac{X_p(i)}{X_u(i)}. && (32)
\end{aligned}$$

We now have n_X+n_B equations in the $2n_B$ unknowns $B_-(j)$ and $B_+(j)$, $j = 1\ldots n_B$. We may eliminate B_+ from the system to reduce the dimensionality:

$$\sum_{j=1}^{n_B}[S_-(i,j) + S_+(i,j)]B_-(j) = \\ A_\mathrm{X}(i)\ n_u^X(i) - n_p(i,0) + \sum_{j=1}^{n_B} S_+(i,j)n_u^B(j),\ \ i = 1\ldots n_X. \quad (33)$$

If we choose to have the same number of i-bins as j-bins, the number n_X of data values is equal to n_B, and Eq. 33 forms a barely-constrained system. The square sub-matrix $S' = S_- + S_+$ that remains after removing the $j = 0$ columns from $S_\pm$ may have an inverse if the problem is sufficiently well-conditioned. However, experience in applying matrix inversion has led to some disappointment, sometimes casting doubt upon this whole approach. A more probable reason for ill-conditioned square matrices is that the experimental data set is simply not adequate to uniquely constrain the Born distributions in such detail. The only alternatives are then to reduce the number of j-bins, or to introduce additional constraints on the results, such as smoothness conditions.

We now develop the solution for the case where S' is a well-conditioned square matrix. Multiplying Eq. 33 by the inverse of S', we get

$$B_-(j) = \sum_{i=1}^{n_X} [S']^{-1}(j,i) \times \left[A_{\mathrm{X}}(i)\ n_u^X(i) - n_p(i,0) + \sum_{k=1}^{n_B} S_+(i,k) n_u^B(k) \right], \ \ j = 1 \ldots n_B. \quad (34)$$

We substitute this solution in

$$A_{\mathrm{Born}}(j) = \frac{2B_-(j) - B_u(j)}{B_u(j)} \quad (35)$$

Again, if the Monte Carlo generator embodies the unpolarized Born cross sections as well as all contaminating processes outside the acceptance,

$$A_{\mathrm{Born}}(j) = -1 + \frac{2}{n_u^B(j)} \sum_{i=1}^{n_X} [S']^{-1}(j,i) \times \left[A_{\mathrm{X}}(i)\ n_u^X(i) - n_p(i,0) + \sum_{k=1}^{n_B} S_+(i,k) n_u^B(k) \right], \quad j = 1 \ldots n_B. \quad (36)$$

As in the simplified case, it's possible to write the absolute statistical uncertainty in A_{Born} in term of a 'radiative dilution matrix':

$$D(j,i) = \frac{2[S']^{-1}(j,i)\ n_u^X(i)}{n_u^B(j)} \quad (37)$$

$$\delta^2(A_{\mathrm{Born}}(j)) = \sum_{i=1}^{n_X} D^2(j,i)\ \delta^2(A_{\mathrm{X}}(i)), \quad (38)$$

where $\delta(A_{\mathrm{X}}(i))$ is calculated as per Eq. 8. We now see uncertainty modification by the matrix D.

Again as in the simplified case, the systematic uncertainty contribution by the beam and/or target polarization (product) P may be inflated by the radiative dilution. To see this, we differentiate Eq. 36 to get

$$|\partial A_{\mathrm{Born}}(j)| = \frac{2}{n_u^B(j)} \sum_{i=1}^{n_X} [S']^{-1}(j,i) A_{\mathrm{X}}(i) n_u^X(i)\ \frac{\partial P}{P^2}$$
$$= \left[A_{\mathrm{Born}}(j)\ + 1 + \frac{2}{n_u^B(j)} \sum_{i=1}^{n_X} [S']^{-1}(j,i) \Big(n_p(i,0) - \sum_{k=1}^{n_B} S_+(i,k) n_u^B(k) \Big) \right] \frac{\partial P}{P}. \quad (39)$$

It was mentioned that statistical precision can be improved if it is assumed that Eq. 26 is independent of i. In this case, all of the subsequent equations still apply if we replace $A_X(i)$ with $\bar{A}_X(i)$ and $n_u^X(i)$ with $\bar{n}_u^X$, where

$$\bar{A}_X(i) = \frac{X_p(i)}{\sum_{k=1}^{n_X} X_u(k)} \tag{40}$$

$$\text{and} \quad \bar{n}_u^X = \sum_{k=1}^{n_X} n_u^X(k). \tag{41}$$

In the general case, matrix inversion may not be applicable. The system may have to be over-constrained ($n_X > n_B$). Then another method must be employed to solve Eq. 28. Typically, one searches the solution spaces for the minimum values of

$$\chi^2 = \left(X_p - [k]S_pB_p\right)^{\mathrm{T}} \left(\mathrm{Cov}_X\right)_p^{-1} \left(X_p - [k]S_pB_p\right). \tag{42}$$

The tool for linear recursion should provide both the solutions and their error matrices. These must then be carried through the calculation of A_{Born} and its uncertainties.

One may be tempted to assume that an estimate of the uncertainty contribution due to MC statistics may be simply obtained by individually varying each of the n matrix elements by one standard deviation and summing in quadrature all of the effects on the result. This approach would be based on the assumption of linear superposition. However, this drastically underestimates the actual uncertainty contribution, which is dominated by the nonlinearity inherent in matrix inversion. It is necessary to vary randomly all of the $n(i,j)$ matrix elements according to their individual expected distributions, in each of many trials, to accumulate a distribution in the resulting A_{Born}. This avoids the cost of many redundant MC runs.

First moments of a Born distribution should be calculated by choosing one large Born bin ($n_B = 1$), to avoid the problem of statistical correlations between smaller bins that must be summed.

An Alternative to a Radiative Monte Carlo Simulation

If a Monte Carlo simulation of the experiment that contains internal radiative effects is not available, it may still be possible to derive the radiative smearing matrices directly from one of the commonly-available codes for calculating local radiative correction factors. By operating this code in a rather

unconventional manner, we may extract the contributions to the observable cross section arising from each element of the Born cross section. The Born cross sections for each spin configuration are perturbed separately, and then the resulting effects on the unpolarized and polarized cross sections are calculated.

A radiative code such as we might employ here takes as input models for the Born cross section, (quasi-)elastic form factors and the quasi-elastic suppression factor. The output is the ratio of radiated to Born cross sections in each bin of experimental kinematics. By observing in all experimental bins the effect on the radiated cross section produced by perturbing the Born cross section in each Born kinematic bin in turn, the correlation matrix S can be computed using the definition Eq. 15. This procedure requires that the binning in Born and experimental kinematics be commensurate — i.e. the Born bin width should be an integral multiple of the experimental bin width. The S matrix must first be computed using the possibly finer experimental binning for the Born bins also, and then a weighted average is computed for each wider final Born bin. It may be necessary to provide the cross section perturbation in each bin as a smoothly varying peaked function, without discontinuities that might corrupt the numerical integrations in the code. In this approach, the B vectors become the actual Born cross sections. The required 'background' contributions $S_{\pm}(i,0)\ \sigma^B_{\pm}(0)$ can be derived as (minus) the effects of setting to zero all Born cross sections outside the region of interest as well as the (quasi-)elastic form factors and the quasi-elastic suppression factor.